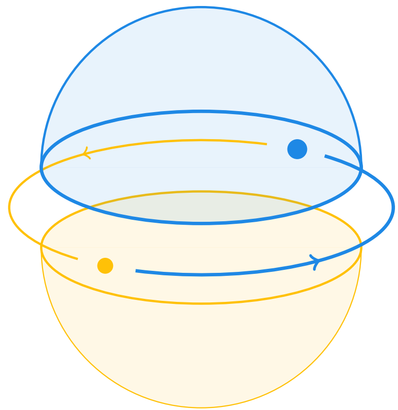
Homotopy theory in the complexity of homomorphism problems

Jakub Opršal et al.*



*based on joint works
with Sebastian Meyer (TU Dresden) and
with Max Hadek (Charles University) and Tomáš Jakl (CTU)

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The constraint satisfaction problem with template A is the problem $\mathrm{CSP}(A)$: Given B, decide if there is a homomorphism $B \to A$.

Theorem [Bulatov, 2017; Zhuk 2017].

For each finite template A either

1. $\mathrm{CSP}(A)$ *is NP-complete, or*
2. $\mathrm{CSP}(A)$ *is in P.*

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Theorem [Bulatov, Jeavons, Krokhin, 2005].
The following are equivalent for finite templates A and B:

1. $\mathrm{CSP}(A)$ *reduces to* $\mathrm{CSP}(B)$ *via a gadget reduction (and hence in log-space);*
2. $\mathrm{pol}(B) \to \mathrm{pol}(A)$.

Theorem [Hell & Nešetřil, 1990].
Let H be a graph. If H is non-bipartite, then H-colouring problem is NP-complete. Otherwise, it is in P.

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[1] Hadek, Jakl, **O** (2025). A **categorical** perspective on constraint satisfaction: The wonderland of adjunctions. *arXiv:2503.10353*.

[2] Meyer, **O** (2025). A **topological** proof of the Hell–Nešetřil dichotomy. *SODA 2025*.

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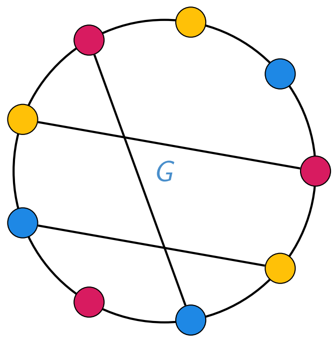
Part I. What problems am I talking about?

Graph colouring



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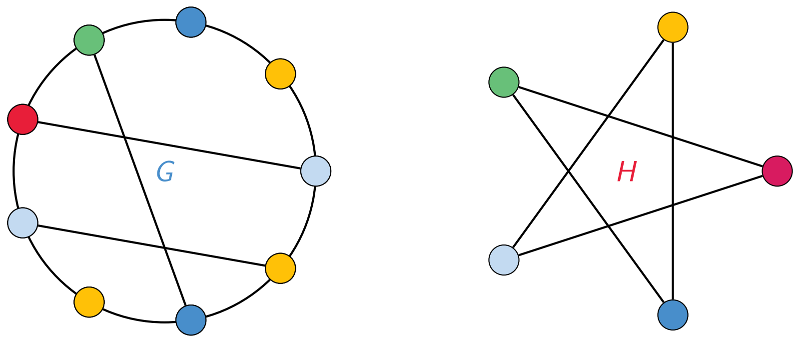

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$uv \in E_G \Rightarrow h(u)h(v) \in E_H.$$

Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.

The H-colouring problem

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H-colouring

Fix a graph H (called *template*). Given a graph G, decide whether there is a homomorphism $G \to H$.

- K_2-colouring is easy (it is solvable in logspace [Reingold, 2005]);
- K_k-colouring is NP-complete for all $k > 2$.
- What about other graphs H?

Theorem [Hell & Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.

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Part II. A proof

Outline of a new proof

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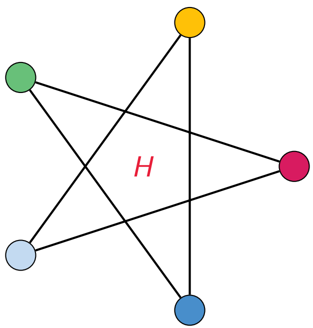

1. Identify which problems are NP-hard using the *algebraic approach to the constraint satisfaction problem*.
2. If H-colouring is not NP-hard, show that its *solution spaces* are component-wise contractible.
3. Use Brower's fixed-point theorem to show that H has a loop if H is not bipartite.

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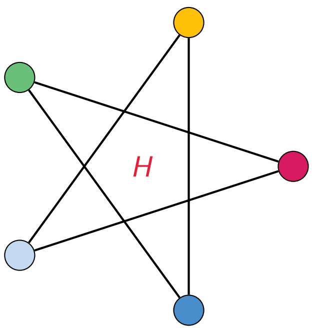

1. Identify which problems are NP-hard using the ~~*algebraic*~~ *categorical approach to the constraint satisfaction problem*.

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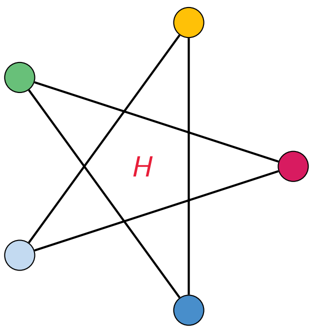

1. **Identify which problems are NP-hard using the ~~*algebraic*~~ *categorical approach to the constraint satisfaction problem*.**
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Categorical approach to the constraint satisfaction problem

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Example.

A (di)graph $G = (V^G, E^G)$ where $E^G \subseteq V^G \times V^G$ can be interpreted as a presheaf $G' \colon \mathcal{E} \to \mathrm{Fin}$ where $\mathcal{E}$ is the category:

$$E \underset{t}{\overset{s}{\rightrightarrows}} V$$

with $G'(V) = V^G$ and $G'(E) = E^G$ where

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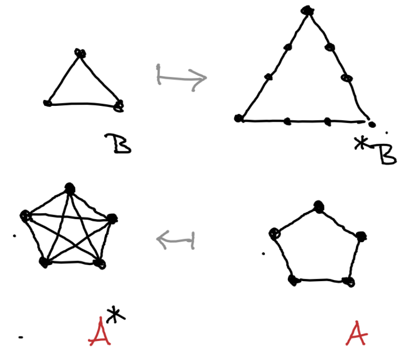
$$\mathrm{CSP}(K_5) \leq_{\text{gadget}} \mathrm{CSP}(C_5)$$



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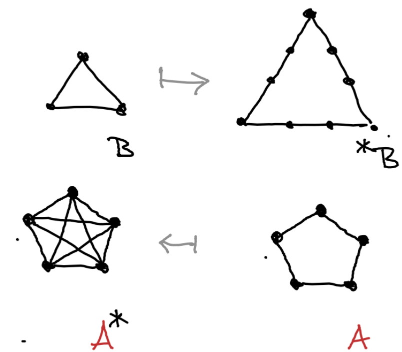

If a functor $L\colon [\mathcal{S}, \mathrm{Fin}] \to [\mathcal{T}, \mathrm{Fin}]$ admits a right adjoint $R\colon [\mathcal{T}, \mathrm{Fin}] \to [\mathcal{S}, \mathrm{Fin}]$, then it is a reduction

$$\mathrm{CSP}(RA) \leq_{\text{gadget}} \mathrm{CSP}(A).$$

Categorical approach to the constraint satisfaction problem (cont.)

Theorem.
$\mathrm{CSP}(A) \leq_{gadget} \mathrm{CSP}(A')$ *iff there is a natural transformation* $\mathrm{pol}(A') \to \mathrm{pol}(A)$.

A polymorphism of A is a homomorphism $f\colon A^n \to A$.

$$\begin{aligned} \mathrm{pol}(A)\colon \mathrm{Fin} &\to \mathrm{Fin} \\ n &\mapsto \{f\colon A^n \to A\} \end{aligned}$$

Lemma.
If $A\colon \mathcal{S} \to \mathrm{Fin}$, *then* $\mathrm{pol}(A) = \mathrm{Ran}_A A$.

$\mathrm{PCSP}(A, B)$: Given C decide between

YES $C \to A$;

NO $C \not\to B$.

Proof.
$\mathrm{CSP}(A) \approx_{\mathrm{gadget}} \mathrm{PCSP}(1_{\mathrm{Fin}}, \mathrm{Ran}_A A)$

$$\begin{tikzcd} {[\mathrm{Fin},\mathrm{Fin}]} \arrow[r, "- \circ A"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & {[\mathcal{S},\mathrm{Fin}]} \arrow[l, bend right=40, "\mathrm{Lan}_A"'] \arrow[l, bend left=40, "\mathrm{Ran}_A"] \end{tikzcd}$$

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- $- \circ A$ is a reduction from PCSP$(1_{\mathsf{Fin}}, \mathrm{Ran}_A A)$ to CSP(A):
  - if $M \to 1_{\mathsf{Fin}}$, then $M \circ A \to 1_{\mathsf{Fin}} \circ A = A$;
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$\mathrm{Lan}_A - \circ A'$ is a reduction from $\mathrm{CSP}(A')$ to $\mathrm{CSP}(A)$ iff $\mathrm{pol}(A') \to \mathrm{pol}(A)$. ∎

Algebraic approach to the constraint satisfaction problem

An operation $t\colon A^n \to A$ is Taylor

$$
\begin{array}{lcl}
t(x \quad * \quad \dots \quad *) & \approx & t(y \quad * \quad \dots \quad *) \\
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Lemma [Taylor, 1977].
If a topological space X admits a continuous idempotent Taylor operation t, then $\pi_1(X)$ is Abelian.

$t\colon A^n \to A$ is *idempotent* if $t(x, \dots, x) \approx x$.

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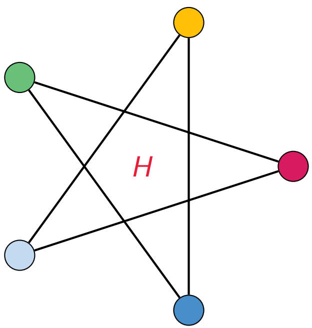

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Intermezzo. What the … is the solution space of H-colouring?

Solution posets: Multihomomorphisms

A multihomomorphism is a function $f: V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ such that, for all edges $uv \in E(G)$, we have that

$$f(u) \times f(v) \subseteq E(H).$$

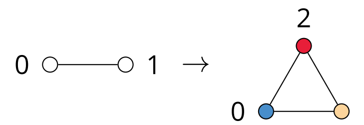
- Multihomomorphisms are naturally ordered

$$f \leq g \Leftrightarrow f(u) \subseteq g(u) \text{ for all } u$$

- $\mathrm{mhom}(G, H)$ is the poset of multihomomorphisms.

Solution posets: Multihomomorphisms

$K_2 \to K_3$



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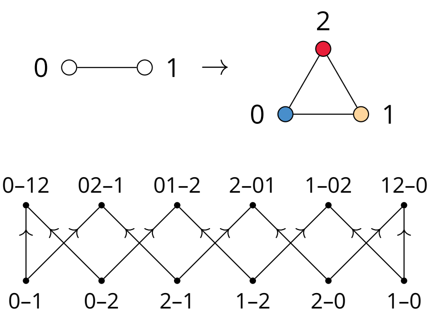

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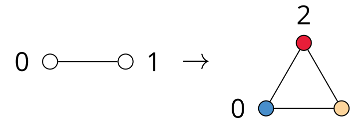
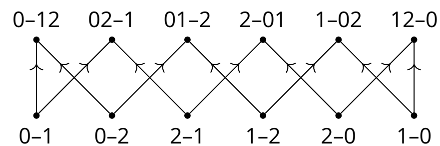
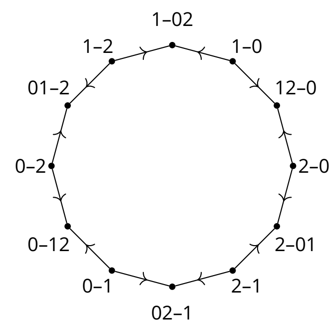






Solution spaces

Given graphs G and H, we define the space

$$\mathrm{Hom}(G, H) = |N\,\mathrm{mhom}(G, H)|$$

- The vertices are multihomomorphisms,
- f and g are connected by an arc if $f \leq g$,
- $\{f, g, h\}$ form a triangle if $f \leq g \leq h$,
- etc.

We view this as the solution space of instance G of H-colouring.

Example. $\mathrm{Hom}(K_2, K_3) \simeq S^1$.

Example. In $\mathrm{mhom}(K_2, K_4)$ we have:

$$0\text{–}1 \leq 02\text{–}1 \leq 02\text{–}13$$

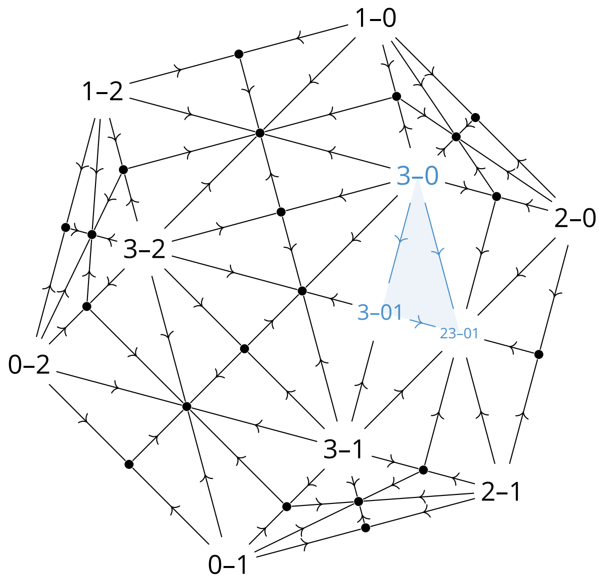
and

$$0\text{–}1 \leq 0\text{–}12 \leq 0\text{–}123$$

which creates 2-dimensional faces.

Two colourings f and g are connected if g can be obtained from f by **changing one value at a time** while remaining a valid colouring.

4-colourings of K_2



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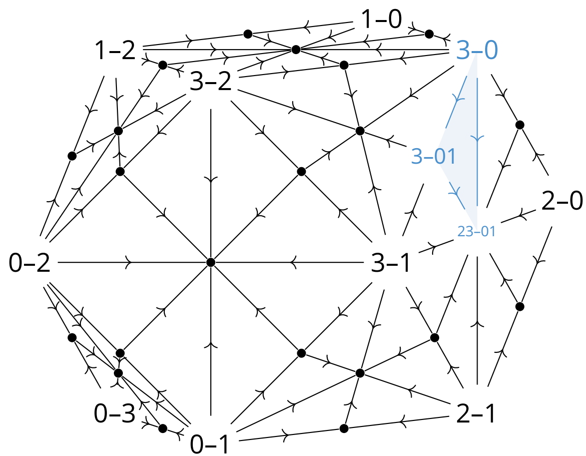

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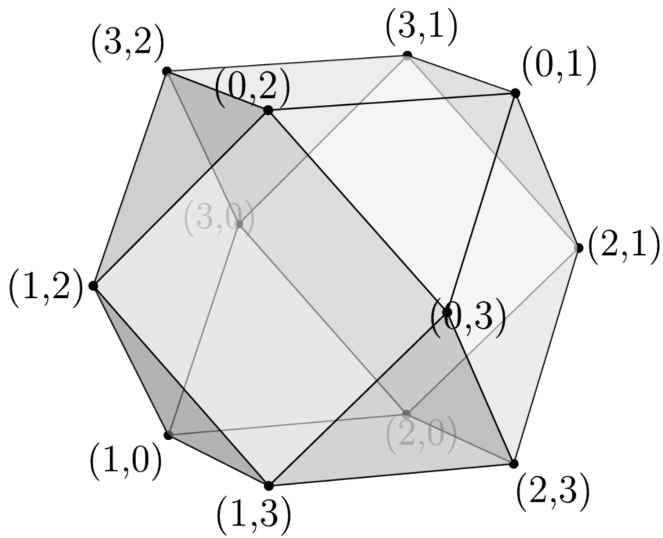

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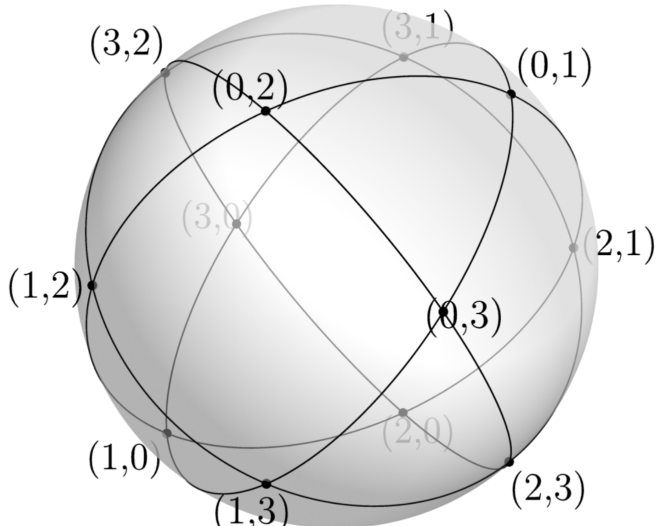

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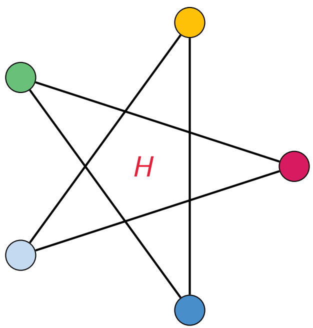
Why is homotopy theory so effective in computational complexity of constraint satisfaction problems?

Part II. A proof (cont.)

Outline of a new proof

Theorem [Hell, Nešetřil, 1990].
Unless P = NP, the only graph H-colouring problem that is solvable in polynomial time is 2-colouring.



1. Identify which problems are NP-hard using the ~~*algebraic*~~ *categorical approach to the constraint satisfaction problem*.

2. **If *H*-colouring is not NP-hard, show that its *solution spaces* are component-wise contractible.**

3. Use Brower's fixed-point theorem to show that *H* has a loop if *H* is not bipartite.

Taylor $\to$ contractibility

A topological space X is called contractible if it is homotopy equivalent to a point $\{*\}$. For us, this is equivalent to $\pi_n(X) = 0$ for all $n \geq 0$.

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Nevertheless, there is a lax-Taylor monotone operation $t\colon \mathrm{mhom}(G, H)^n \to \mathrm{mhom}(G, H)$ that satisfies:

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\begin{array}{c}
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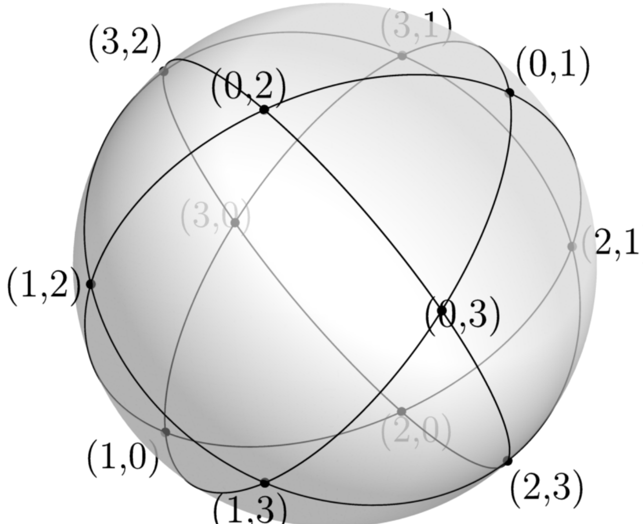
Theorem [Meyer, **O**, 2025].
Every connected finite poset that admits a monotone lax-Taylor operation is contractible, and therefore $\mathrm{Hom}(G, H)$ *is component-wise contractible for all G if H has a Taylor polymorphism.*

4-colourings of K_2



Hence, 4-colouring is NP-hard!

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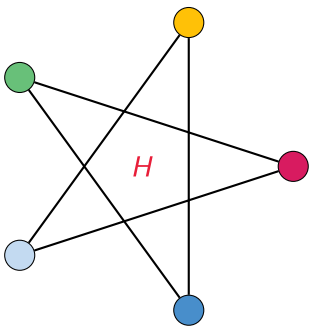

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Every continuous function $f : D^n \to D^n$ *has a fixed point, i.e., there exists* $x \in D^n$ *such that* $f(x) = x$.

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More generally: If X is a contractible compact CW-complex, then every function $f : X \to X$ has a fixed point.

A $\mathbb{Z}_2$-action on $\mathrm{Hom}(K_2, H)$

The space $\mathrm{Hom}(K_2, H)$ admits an action of the group $\mathbb{Z}_2$, i.e., there is a homeomorphism

$$\phi\colon\ \mathrm{Hom}(K_2, H) \to \mathrm{Hom}(K_2, H)$$

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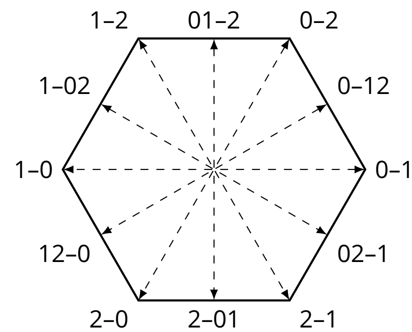

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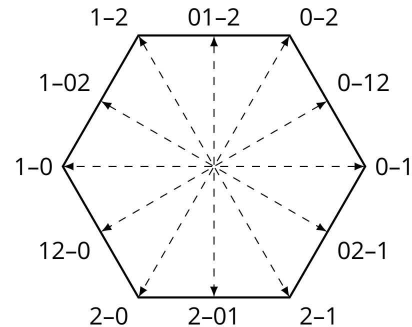

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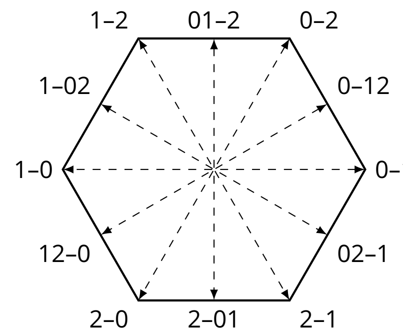

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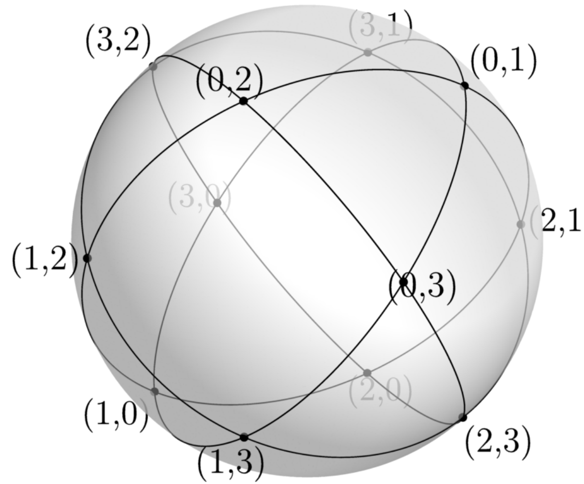

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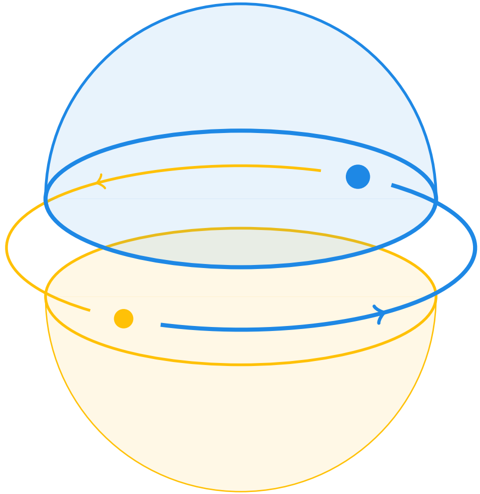

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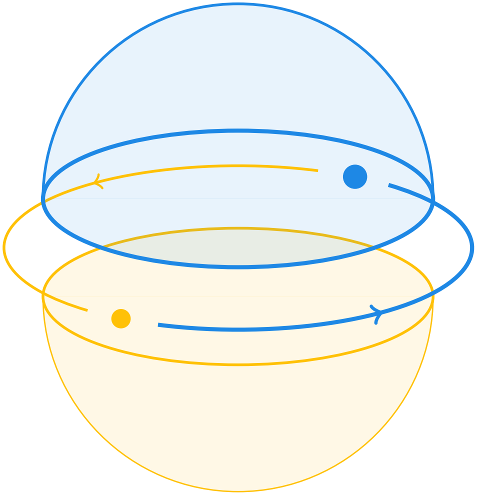
- ϕ is defined by flipping the two values of each multihomomorphism.
- flipping the two values induces a monotone involution on the poset, and hence a continuous involution on the space.
- ϕ does not have a fixed point, otherwise H would contain an edge uu.



The proof

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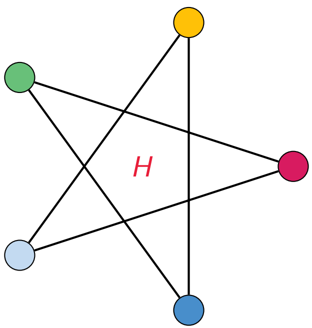

Proof. Assume that H is not-bipartite, and consider the space $X = \operatorname{Hom}(K_2, H)$.

Observe that the space admits a fixed-point free $\mathbb{Z}_2$-action $\phi\colon X \to X$ that for each multihomomorphism m flips the values of $m(0)$ and $m(1)$.

If H is not-bipartite then ϕ fixes a connected component of X. Indeed, if uv is an edge of an odd cycle of H then uv is connected to $vu = \phi(uv)$.

If H admitted a Taylor homomorphism, $\operatorname{mhom}(K_2, H)$ would admit a lax-Taylor operation, and all its connected component would be contractible.

Hence, ϕ which acts on the component of uv has a fixed point, the contradiction. ∎

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How it started...

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Conjecture [Brakensiek, Guruswami, 2018].

Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ with c colours, i.e., $\mathrm{CSP}(C_{(2k+1)}, K_c)$*, is NP-complete for all* $c > 2$ *and* $k > 0$.

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