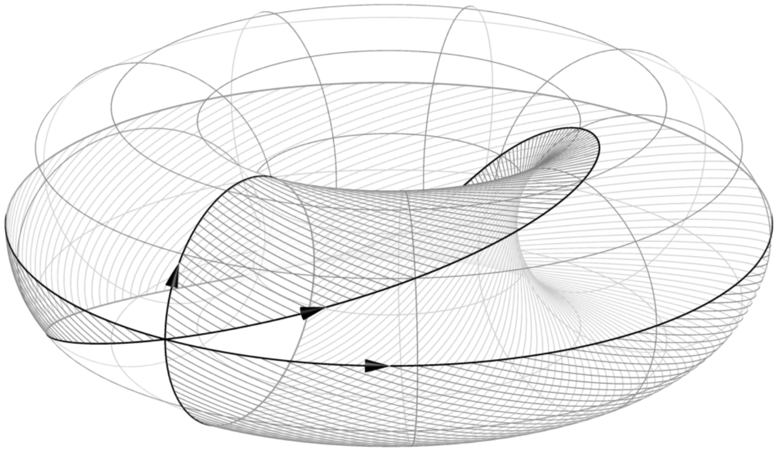
NP-hardness of promise colouring graphs via homotopy

Jakub Opršal et al.*



* A. Krokhin, M. Wrochna, S. Živný;
M. Filakovský, T-V. Nakajima, G. Tasinato, U. Wagner

UNIVERSITY OF BIRMINGHAM

Is there a polynomial time algorithm that colours a given 3-colourable graph by 3 4 5 6 7 1729 2^{15} $O(\log n)$ $\tilde{O}(n^{0.19747})$ colours?

No! (Unless P $=$ NP) [Karp, 1972] **No!** (Unless P $=$ NP) [Khanna, Linial, Safra, 2000] (We didn't know.) (We don't know.) **Yes!** [Kawarabayashi, Thorup, Yoneda, 2024]

A black box: Algebraic approach

Theorem [Barto, Bulín, Krokhin, O., '21].

Let Γ and Δ be two promise CSPs. If there is a natural transformation $\mathrm{pol}(\Delta) \to \mathrm{pol}(\Gamma)$, then there is a log-space reduction from Γ to Δ.

$\mathrm{pol}(?) \colon \mathbf{set}_{<\aleph_0} \to \mathbf{set}_{<\aleph_0}$ *denotes the polymorphism minion of the problem ?.*

There is a natural transformation $\mathrm{pol}(\Delta) \to \mathrm{pol}(\Gamma)$ where

- Δ is the problem of 27480-colouring 2-colourable 3-uniform hypergraphs, which was proven to by NP-hard by [Dinur, Regev, Smyth, '05].
- Γ is 5-colouring 3-colourable graphs.

Corollary [Barto, Bulín, Krokhin, O., '21].

Colouring graphs that are promised to be 3-colourable with 5 colours is NP-hard. ∎

Chapter I: The story begins...

If colouring 3-colourable graphs with 3-colours is NP-hard, can we give a stronger promise to make the problem easier?

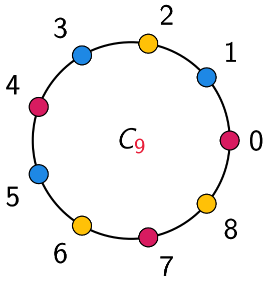
Andrei Krokhin & O.

What stronger promise can we give?

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a graph homomorphism $G \to H$ is a mapping $h\colon V_G \to V_H$ that preserves edges,

$$(u, v) \in E_G \Rightarrow (h(u), h(v)) \in E_H.$$

Example. A colouring of a graph G with k colours is just a homomorphism $c\colon G \to K_k$.



Promise that the input graph maps to an odd cycle C_{2k+1}!

The problems

Promise graph homomorphism PCSP(C_{2k+1}, K_3).

Given a graph G that is promised to map to C_{2k+1}, find a 3-colouring:

$$G \to C_{2k+1} \Rightarrow c\colon G \to K_3$$

Promise hypergraph homomorphism PCSP(LO_3, LO_4).

Given a 3-uniform hypergraph H that is promised to be linearly-ordered 3-colourable, find a linearly-ordered 4-colouring:

$$H \to LO_3 \Rightarrow c\colon H \to LO_4$$

Linearly-ordered hypergraph colouring is mapping $c\colon H \to [k]$ such that for each $(u_1, u_2, u_3) \in E^H$, there exists $i \in \{1, 2, 3\}$ such that $c(u_i) > c(u_j)$ for all $j \neq i$.

Graph colouring and its polymorphisms

The problem PCSP(C_{2k+1}, K_3).

Given a graph G that is promised to map to C_{2k+1}, find a 3-colouring:

$$c\colon G \to K_3$$

Definition.

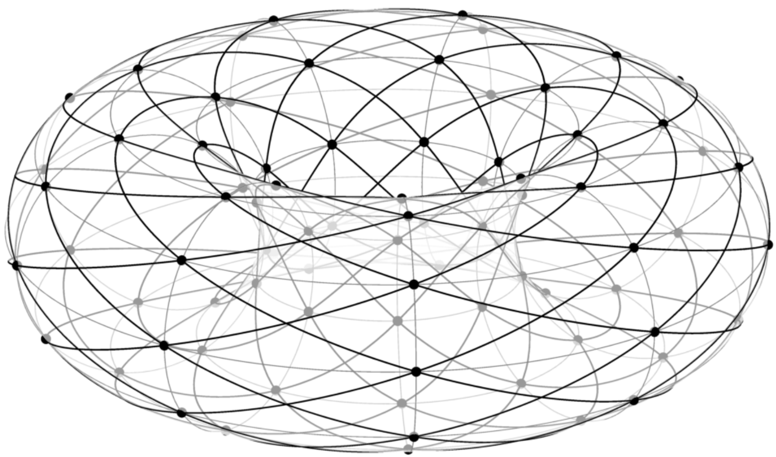
A polymorphism of PCSP(C_{2k+1}, K_3) is a homomorphism $f\colon C_{2k+1}^n \to K_3$, i.e., a mapping $f\colon [2k+1]^n \to \{\bullet, \bullet, \bullet\}$ such that

$$(f(u_1, \dots, u_n), f(v_1, \dots, v_n)) \in E_{K_3}$$

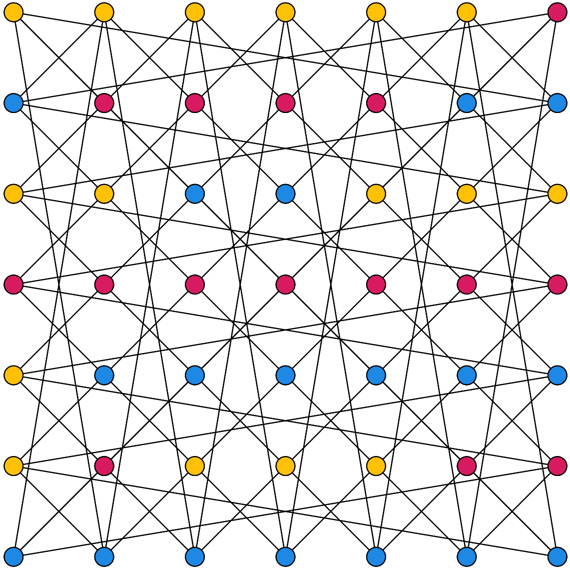
whenever $(u_i, v_i) \in E_{C_{2k+1}}$ for all $i \in [n]$.

$\mathrm{pol}(C_{2k+1}, K_3)\colon [n] \mapsto \mathrm{hom}(C_{2k+1}^n, K_3)$

$C_9{}^2$



Polymorphisms ${C_7}^2 \to K_3$



Two continuous functions $f, g\colon X \to Y$ are said to be

homotopic

if is there is a continuous function $H\colon X \times [0, 1] \to Y$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$.

From graphs to topological spaces

Fix a test graph $T = K_2$. A multimorphism from T to G is a mapping $f \colon V^T \to \mathscr{P}(V^G) \setminus \{\emptyset\}$ such that

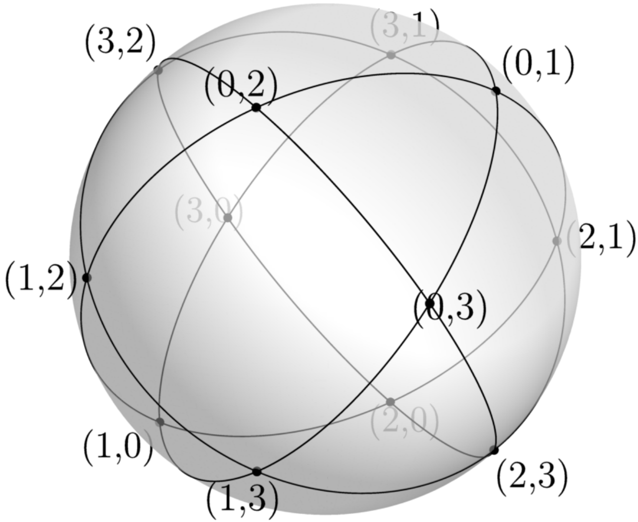
$$(u, v) \in E^T \Rightarrow f(u) \times f(v) \subseteq E^G$$

We denote the poset of all such multimorphisms by $\mathrm{mhom}(T, G)$.
Multimorphisms are ordered by $f \leq g \coloneqq \forall x (f(x) \subseteq g(x))$.

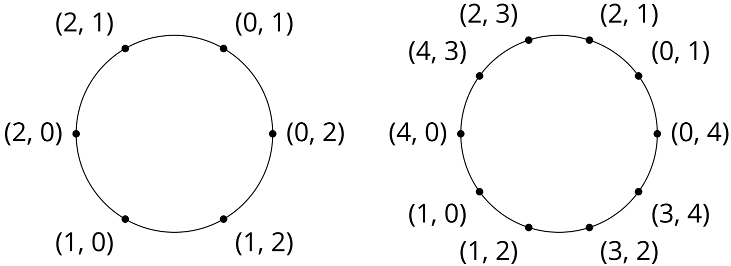
Definition.
The topological space $\mathrm{Bx}(G)$ is the geometric realisation of the nerve of $\mathrm{mhom}(K_2, G)$.

Two graph homomorphisms f and g are homotopic if g can be obtained from f by **changing one value at a time** while remaining a valid homomoprhism.

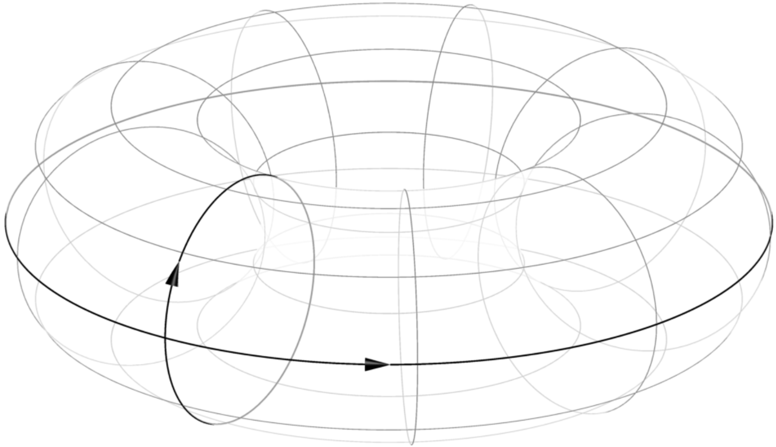
Bx(K_4)



$\mathrm{Bx}(K_3)$, $\mathrm{Bx}(C_5)$, ...



$\mathrm{Bx}(C_{2k+1}{}^2)$*



*up to homotopy equivalence

A natural transformation

- There is a natural (in n) map:

$$\hom({C_{2k+1}}^n, K_3) \to [T^n, S^1]$$

- Homotopy classes of continuous maps $T^n \to S^1$ are in 1-to-1 correspondence with linear maps

$$\mathbb{Z}^n \to \mathbb{Z}$$

(since $S^1 = B\mathbb{Z}$ and $H^1(T^n, \mathbb{Z}) \simeq \mathbb{Z}^n$).

$$T^n = (S^1)^n = S^1 \times \cdots \times S^1$$
$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

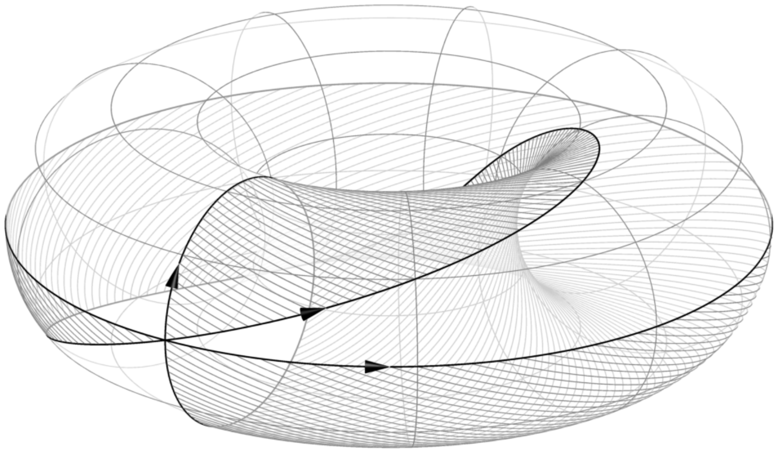
Altogether, we get a natural transformation:

$$\xi\colon \ \mathrm{pol}(C_{2k+1}, K_3) \to \mathrm{pol}(\mathbb{Z}).$$

Corollary [Barto, Bulín, Krokhin, O., '21].

Let Γ be a finite template promise CSP. If there is a natural transformation $\xi\colon \ \mathrm{pol}(\Gamma) \to \mathrm{pol}(\mathbb{Z})$ such that $\xi(f) \neq 0$ for all $f \in \mathrm{pol}(\Gamma)$, then Γ is NP-complete.

A minion homomorphism



Epilogue

Theorem [Krokhin, O., '19].
Colouring graphs that are promised to map homomorphically to $C_{(2k+1)}$ *with 3 colours is NP-hard.*

*the proof was brought to you by [Wrochna, Živný, '20]

Krokhin, O., Wrochna, & Živný. (2023). Topology and adjunction in promise constraint satisfaction. *SIAM Journal on Computing*, 52(1), 38–79. arXiv:2003.11351, doi:10.1137/20M1378223

Theorem [Filakovský, Nakajima, O., Tasinato, Wagner, STACS'24].
Linearly ordered 4-colouring of 3-colourable 3-uniform hypergraphs is NP-hard.

Filakovský, Nakajima, O., Tasinato, & Wagner. (2024). Hardness of linearly ordered 4-colouring of 3-colourable 3-uniform hypergraphs. *Symposium on Theoretical Aspects of Computer Science, STACS 2024*, (pp. 34:1–34:19). arXiv:2312.12981, doi:10.4230/LIPIcs.STACS.2024.34