A categorical perspective on constraint satisfaction

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CSPs

SAT

HornSAT

graph colouring

3-colouring

2-colouring

2SAT

$$\begin{cases} x+y+z=1 \\ y+z+u=0 \\ x+z+u=1 \end{cases}$$

st-connectivity

We will treat CSPs as the homomorphism problem:

CSP(A)

Fix A (template).
Given B (instance)
Decide:

$\exists ?\, f : B \to A$

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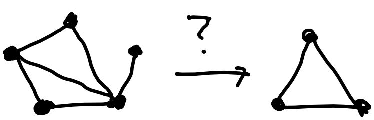

$CSP(K_3) \equiv$ 3-colouring

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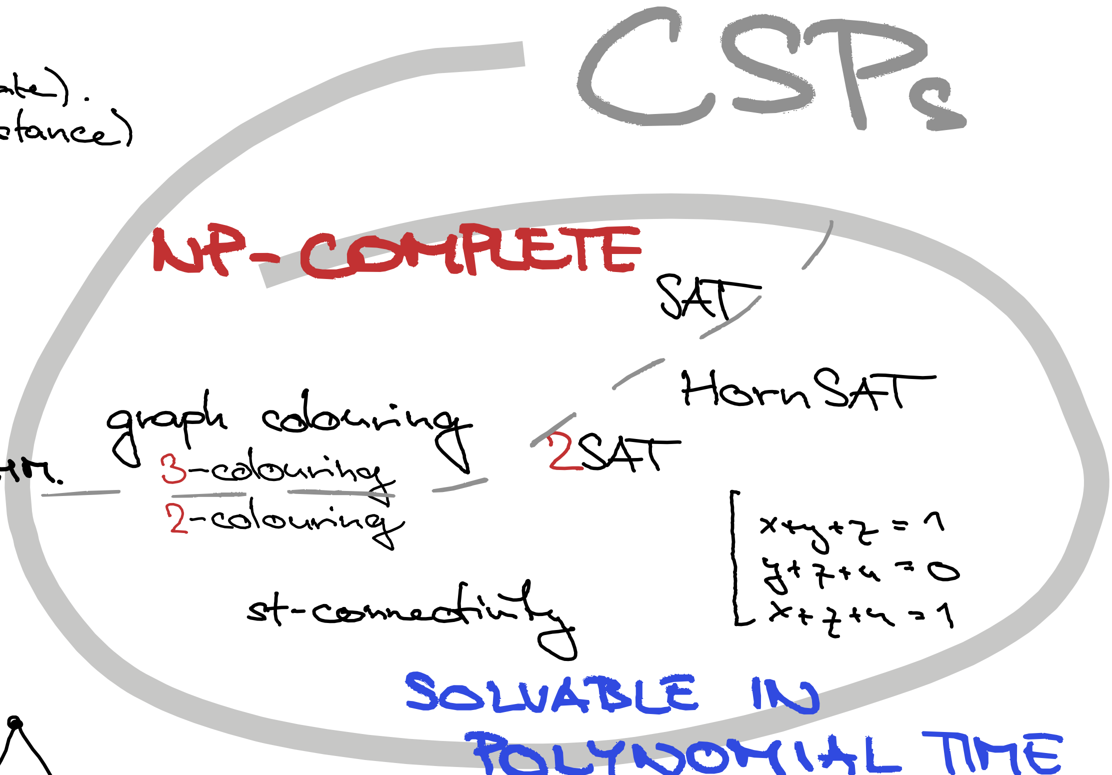
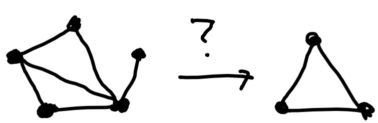


BULATOV–ZHUK THM.
CSP(A) ∈ NP-c ∪ P
(A finite)



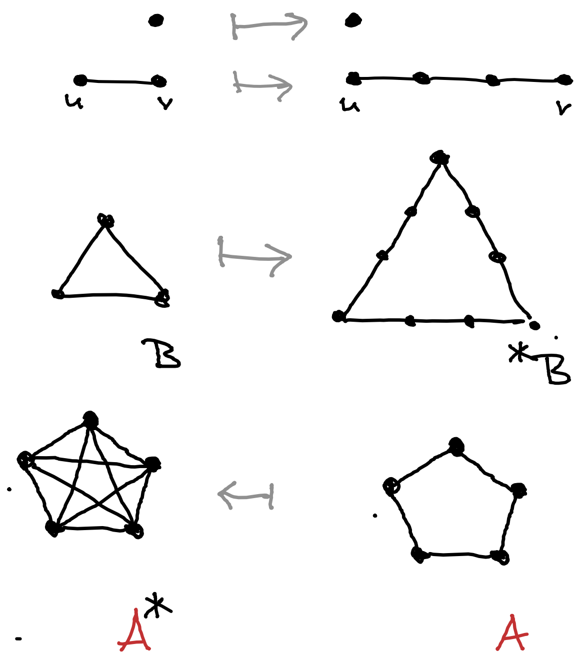
$CSP(K_3) \equiv$ 3-colouring

CSP(A) given B

decide if $\exists f: B \to A$

REDUCTIONS:

from CSP(K_5) to CSP(C_5)



[Hell-Nešetřil, 1990]

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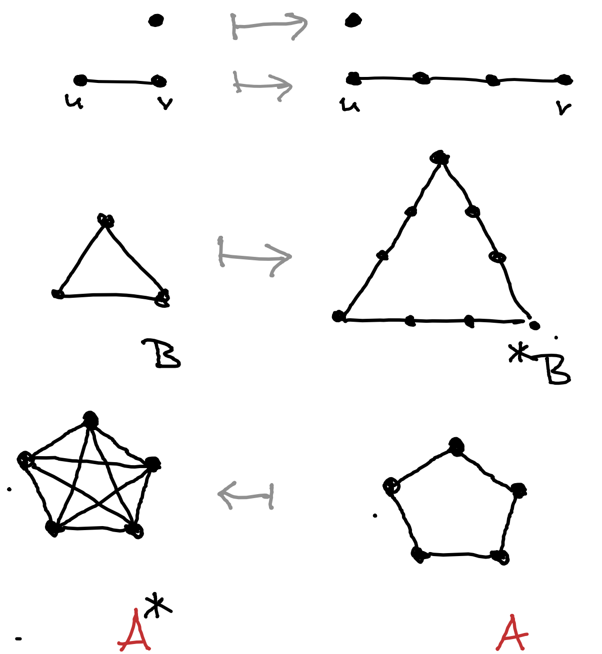


[Hell-Nešetřil, 1990]

Graphs as presheaves over:

$$\mathcal{C} = \left(E \leftleftarrows V \right)$$

To reduce from CSP(A') to CSP(A) where:

- $A : \mathcal{C} \to \mathrm{Fin}$, $A' : \mathcal{C}' \to \mathrm{Fin}$, and
- $\mathcal{C}, \mathcal{C}'$ are finite categories,

we use a profunctor

$$G : \mathcal{C}' \times \mathcal{C}^{op} \to \mathrm{Fin}$$

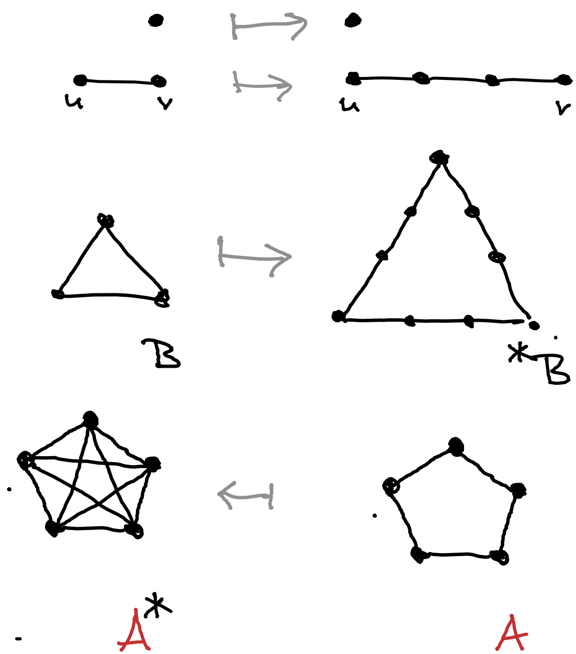
$$\begin{tikzcd} {[\mathcal{C}', \mathrm{Fin}]} \arrow[r, bend left, "G \otimes -"] \arrow[r, phantom, "\perp"] & {[\mathcal{C}, \mathrm{Fin}]} \arrow[l, bend left, "N_G"] \end{tikzcd}$$

s.t. $N_G(A) \rightleftarrows A'$

$\mathrm{CSP}(A)$ given B
decide if $\exists f: B \to A$

REDUCTIONS:

from $\mathrm{CSP}(K_5)$ to $\mathrm{CSP}(C_5)$



[Hell–Nešetřil, 1990]

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WHEN DOES SUCH G EXIST?

THEOREM

Up to these reductions, each $\mathrm{CSP}(A)$ is equivalent to a unique problem of the form

$$\mathrm{CSP}(\mathrm{Id}, M)$$

where $\mathrm{Id}, M : \mathrm{Fin} \to \mathrm{Fin}$.

$\mathrm{CSP}(\mathrm{Id}, M)$ is a promise problem:

yes if $X \to \mathrm{Id}$
no if $X \not\to M$

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PF.

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2) from $\mathrm{CSP}(\mathrm{Id}, M)$ to $\mathrm{CSP}(A)$

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$$\begin{tikzcd}
{[\mathrm{Fin}, \mathrm{Fin}]} \arrow[rr, "- \circ A" description] & & {[\mathcal{S}^{\mathrm{op}}, \mathrm{Fin}]} \arrow[ll, bend right=50, "\mathrm{Lan}_A"'] \arrow[ll, bend left=50, "\mathrm{Ran}_A"] \arrow[ll, phantom, shift right=4, "\perp"] \arrow[ll, phantom, shift left=4, "\perp"]
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$$\boxed{M = \mathrm{Ran}_A A}$$

$\square$

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ANSWER: $\# \; \mathrm{Ran}_A A \longrightarrow \mathrm{Ran}_{A'} A'$

THEOREM [Bulatov, Jeavons, Krokhin; 2005]

- If $\mathrm{Ran}_{A} A \longrightarrow \mathrm{Ran}_{A'} A'$, then $\mathrm{CSP}(A')$ reduces to $\mathrm{CSP}(A)$ in logspace.
- $\mathrm{CSP}(A)$ is NP-complete if $\mathrm{Ran}_{A} A \longrightarrow \mathrm{Id}$.

We only need a pair of functors L, R such that

$$\exists f \in \hom(L(A), B) \text{ iff } \exists g \in \hom(A, R(B))$$

QUESTION

Can we similarly characterise a wider class of such functors?

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Jakl, Hadek, Opršal, A categorical perspective on constraint satisfaction: The wonderland of adjunctions, arXiv: 2503.10353

Thank you!